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TOPOLOGICAL REPRESENTATION OF SEMIGROUPS

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1. Introduction

J. de Groot has proved in [3] that for every group G one can find a connected metric space M such that the group of all autohomeomorphisms of M is isomorphic to G : $G \simeq A(M)$.

To represent semigroups in a similar way, we must replace the group of autohomeomorphisms by a suitable semigroup of continuous mappings. The aim of this note is to prove that every semigroup S with identity element can be represented by the semigroup $Q(M)$ of all quasi-local homeomorphisms of a metric space M into itself.

Let X, Y be topological spaces. A mapping $f: X \rightarrow Y$ is called a *quasilocal homeomorphism* if f is continuous and if for each open set $O \subset X$ there exists an open set $V, V \subset O$ such that $f|V$ is a homeomorphism of V onto $f(V)$.

The proof of the theorem is essentially a modification of the proof for groups by J. de Groot in [3].

The semigroup $Q(M)$ of all quasi-local homeomorphisms seems to be the most suitable to replace the group of all autohomeomorphisms $A(M)$. We prove in section 4 the existence of a semigroup S such that there is no Hausdorff-space H such that S is isomorphic to the semigroup of all local homeomorphisms of H into itself. Neither can S be isomorphic to the semigroup of all open continuous mappings of H into itself. $f: X \rightarrow Y$ is a local homeomorphism if for each $x \in X$ there exists an open set $O, x \in O$ such that $f|O$ is a homeomorphism of O onto $f(O)$.

Analogous problems were treated by Z. Hedrlín and A. Pultr [6] and by L. Bukovský, Z. Hedrlín and A. Pultr [1]. In [6] the following theorem was proved. Let S be a semigroup with identity element, then there exists a T_0 -space T such that S is isomorphic to the semigroup of all local homeomorphisms of T into itself.

In [1] it has been shown that every semigroup with identity element may be represented by the semigroup of all "quasi-coverings" of a Hausdorff space into itself. The "quasi-coverings" however are rather special mappings.

Let for instance X be the subset of the real line R consisting of the point 0 and all $x, x \geq 1$. $X = \{x \mid x \in R, x = 0 \text{ or } x \geq 1\}$.

Let $f: X \rightarrow X$ and $g: X \rightarrow X$ be defined respectively by

$$f(x) = \begin{cases} x & \text{if } x = 0 \\ 2x & \text{if } x \neq 0 \end{cases}, \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } x \neq 0 \end{cases}.$$

Both f and g are homeomorphisms of X into X , f however is a quasicovering of $f(X)$ but g is not a quasi-covering of $g(X)$.

2. Graph-representations

Let S be a semigroup with identity element e and $\{s_\alpha\}$ a system of generators of S . We now construct the Cayley-graph S' of S . S' is a coloured, directed graph such that each element $a \in S$ is represented by one vertex v_a of S' . Two vertices v_a and v_b are joined by an edge with "colour" s_α directed from v_a to v_b whenever $b = s_\alpha a$. S' is clearly connected (if $a = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_n}$, then v_e and v_a are joined by a path along a set of consecutively adjacent edges with colour respectively $s_{\alpha_n}, s_{\alpha_{n-1}}, \dots, s_{\alpha_2}, s_{\alpha_1}$). With each $a \in S$ we associate the inner right translation ϱ_a

$$\varrho_a : x \rightarrow xa \quad \text{for all } x \in S.$$

When applying products of mappings from the left to the right

$$(x) \varrho_a \cdot \varrho_b = (x \varrho_a) \varrho_b$$

we see that S is homeomorphic to its regular representation S_r . This representation is faithful since S contains an identity element: $S \simeq S_r$. Furthermore it can easily be seen that S_r is isomorphic to the semigroup of all transformations of the graph S' into itself which are colour and orientation preserving.

If S is a semigroup with cancellation then all such transformations are one-to-one mappings of S' into itself.

From S' we now construct an (un coloured) directed graph S^* such that the semigroup of all endomorphisms $E(S^*)$ of S^* is isomorphic to S . For countable semigroups this has been done first by the author [7], for semigroups with cardinality less than the first unaccessible cardinal by Z. Hedrlín and A. Pultr [5] and for arbitrary semigroups by P. Vopěnka, A. Pultr and Z. Hedrlín [8]. They constructed for any cardinal m a directed graph X such that the identity transformation is the only endomorphism of X and such that the cardinal of the set of vertices of X is equal to m .

The construction of S^* given here is different from the one in [5], since the rigid graph X plays a completely different role.

Construction. Let S' be the Cayley-graph of S and let m be the cardinal of the set of generators $\{s_\alpha\}$ of S . We assume $m \geq 3$ (the case of semigroups of order < 3 can be treated separately in a simple way). Let D be the rigid graph constructed in [8], where $D = \{\beta \mid \beta \leq \omega_\xi + 1, \omega_\xi \text{ the least ordinal with card } \omega_\xi = m\}$. Finally let ϕ be a one-to-one mapping of the set $\{s_\alpha\}$ onto D .

Suppose that a directed edge with colour s_α leads from vertex v_a to v_b . Replace the edge in S' by a graph (D, α, a, b) defined as follows: edges $(v_a, p_{a,b}^\alpha), (p_{a,b}^\alpha, v_b)$,

$(p_{a,b}^\alpha, \phi(s_\alpha))$ and furthermore D . We do this for every edge of S' , but we take care that all graphs (D, α, a, b) are disjoint with the possible exception of their vertices v_a and v_b . In this way S' is transformed into a graph S^* .

Theorem 1. $E(S^*) \simeq S$.

Proof. Let $f \in E(S^*)$ and let $D_{a,b}^\alpha$ be the copy of D contained in the subgraph (D, α, a, b) of S^* .

We first prove that $f(D_{a,b}^\alpha) \subset D_{c,d}^\gamma$ for some γ, c and d .

Since $D_{a,b}^\alpha$ contains the edges

$$(0_{a,b}^\alpha, 1_{a,b}^\alpha), (0_{a,b}^\alpha, 2_{a,b}^\alpha) \quad \text{and} \quad (1_{a,b}^\alpha, 2_{a,b}^\alpha)$$

it follows that $f(0_{a,b}^\alpha)$ cannot be a vertex of the form v_a or $p_{a,b}^\alpha$ of S^* . Hence $f(0_{a,b}^\alpha) \subset D_{c,d}^\gamma$ for some γ, c and d .

If $\beta_{a,b}^\alpha \in D_{a,b}^\alpha$, then there is a finite chain of directed edges connecting $0_{a,b}^\alpha$ and $\beta_{a,b}^\alpha$. From this it follows that $f(\beta_{a,b}^\alpha) \in D_{c,d}^\gamma$, hence $f(D_{a,b}^\alpha) \subset D_{c,d}^\gamma$.

From the rigidity of D it follows that $f(\beta_{a,b}^\alpha) = \beta_{c,d}^\gamma$.

We next prove that $f(p_{a,b}^\alpha) = p_{c,d}^\gamma$.

Since $p_{a,b}^\alpha$ is connected with $\phi(s_\alpha)_{a,b}^\alpha$, we have $f(p_{a,b}^\alpha) = p_{c,d}^\gamma$ which implies $\gamma = \alpha$ or $f(p_{a,b}^\alpha) \in D_{c,d}^\gamma$.

In this case $f(p_{a,b}^\alpha) = \beta_{c,d}^\gamma$ for some $\beta \in D$ $\beta < \phi(s_\alpha)$. Now let α' be chosen so that $\phi(s_{\alpha'}) = \beta$, and let $q = s_{\alpha'} \cdot b$. Then it follows from the construction of S^* that $f(v_b) \in D_{c,d}^\gamma$, hence $f(p_{b,q}^{\alpha'}) \in D_{c,d}^\gamma$ and this implies $f(\phi(s_{\alpha'})_{b,q}^{\alpha'}) = \phi(s_{\alpha'})_{c,d}^\gamma \in D_{c,d}^\gamma$.

From the construction of D it then follows that $\beta < \phi(s_{\alpha'})$ a contradiction.

Thus each vertex of the form $p_{a,b}^\alpha$ of S^* is mapped onto a vertex of the form $p_{c,d}^\gamma$. From this it follows that each vertex of the form v_a is mapped onto a vertex of the form v_b .

It can now easily be seen that $E(S^*)$ is isomorphic to the semigroup of all transformations of S' into itself which are colour and orientation preserving. Hence $E(S^*) \simeq S$.

If S is a semigroup with cancellation then each transformation $f \in E(S^*)$ is one-to-one.

3. Quasi-local homeomorphisms

Similarly as in [3] we shall replace every edge of S^* by mutually homeomorphic topological spaces P and introduce a topology in the resulting set such that a space M will be obtained satisfying the following condition:

$$Q(M) \simeq S.$$

An example of a Peano curve P which is rigid under topological transformations of P into P was given in [2]. We briefly mention its construction.

Consider a circle C^1 in the plane and let $\{a_i^k\}_{i,k}$ be a double sequence of distinct natural numbers > 2 . Let $\{p_i^1\}$ be a countable everywhere dense subset of C^1 . Affixe to each p_i^1 a chain C_i^1 of a_i^1 links, contained in the interior of C^1 (p_i^1 excepted) and mutually disjoint. Next we take a countable dense subset $\{p_i^2\}$ on the union of all C_i^1 such that each p_i^2 is of order two. Affixe to each p_i^2 a chain C_i^2 of a_i^2 links contained in the interior of that link to which p_i^2 belongs, and such that all new chains are mutually disjoint. Proceed by induction; we take care that the diameters of the C_i^k tend to zero, and take the closure P of the countable number of chains obtained in this manner. We remark that P is not rigid for topological transformations of P into P only, but also for quasi-local homeomorphisms.

Let f be a quasi-local homeomorphism and let $\{p_i^k\}^*$ be the set of all points p_i^k such that there is an open set O , $p_i^k \in O$ with $f|_O$ a homeomorphism. The set $\{p_i^k\}^*$ is everywhere dense in P . Since the p_i^k are the only points of maximal order (order 6) in P , the set $\{p_i^k\}^*$ is mapped into the set $\{p_i^k\}$. To each p_i^k is affixed a chain of a_i^k links, all a_i^k distinct. This implies that $f(p_i^k) = p_i^k$ for all $p_i^k \in \{p_i^k\}^*$. Since $\{p_i^k\}^*$ is dense in P , f is the identity transformation.

Now let a and b be two points on the circle C^1 of order two. Each directed edge $\alpha = (\overrightarrow{x_1, x_2})$ of S^* is replaced by a copy P_α of P , a replacing x_1 and b replacing x_2 . We take care that all P_α are disjoint with the possible exception of the points a and b .

Into the union of all P

$$M = \bigcup_{\alpha} P_{\alpha}$$

we introduce a metric in the same way as in [3].

Theorem 2. *Let S be a semigroup with identity element. Then there exists a connected metric space M such that S is isomorphic to the semigroup of all quasi-local homeomorphisms of M : $S \simeq Q(M)$.*

Proof. Let M be the metric space, obtained from the graph S^* . M is clearly connected.

If $f^* \in E(S^*)$, then it can easily be seen that f^* can be extended to a quasi-local homeomorphism f of M into M .

Now let f be a quasi-local homeomorphism of M into M . We shall prove that f maps every copy of P identically onto a copy of P . Let P_α be such a copy of P . P_α is compact and connected, hence $f(P_\alpha)$ is compact, which implies $f(P_\alpha) \subset \bigcup_{i=1}^n P_{\beta_i}$. Let $\{p_i^k\}^*$ be the set of all points $p_i^k \in P_\alpha$ such that there is an open set O , $p_i^k \in O$ with $f|_O$ a homeomorphism. Then $\{p_i^k\}^*$ is mapped into the set of all points of maximal order in $\bigcup_{i=1}^n P_{\beta_i}$ together with the set of endpoints $\{a_{\beta_i}, b_{\beta_i}\}_{i=1}^n$.

Let $\{p_i^k\}^1 \subset \{p_i^k\}^*$ be the set of all points which are mapped into the set of all points of maximal order in $\bigcup_{i=1}^n P_{\beta_i}$. Then $\{p_i^k\}^1$ is everywhere dense in P_α , and it is not difficult to see that each point $p_i^k \in \{p_i^k\}^1$ is mapped onto the corresponding point p_i^k contained in one of the P_{β_i} . From this it follows that every point $x \in P_\alpha$ is mapped onto a corresponding point x contained in one of the P_{β_i} .

Since we have chosen the endpoints a and b of P to be points of order two and since S^* contains no trivial cycles of order two it follows that P_α is mapped identically on another copy P_β of P .

Hence f permutes the P_α 's among themselves, and we may conclude from theorem 1 that $S \simeq E(S^*) \simeq Q(M)$.

Corollary. *Let S be a semigroup with cancellation, with identity element. Then there is a connected metric space M such that S is isomorphic to the semigroup of all homeomorphisms of M into M .*

The proof follows easily from the fact that in this case each transformation $f^* \in E(S^*)$ is one-to-one.

Theorem 3. *Let S be a semigroup with identity element. Then there exists a connected compact Hausdorff space H such that S is isomorphic to $Q(H)$.*

Proof. Let M be the metric space such that $S \simeq Q(M)$, and let H be the Čech-Stone compactification of M . Let f be a quasi-local homeomorphism of M into M and βf its extension to H . Since M contains an open dense subset such that every point of this set has a neighbourhood with compact closure, it follows that for every open set $O \subset H$ there is an open set V , $V \subset O$ such that $V \subset M$. This together with the fact that βf is continuous implies that βf is a quasi-local homeomorphism of H .

Now let g be an element of $Q(H)$. As g is a quasi-local homeomorphism there is for every open set $O \subset H$ an open set $V \subset M$ such that $g|_V$ is a homeomorphism.

Since M is metric, it satisfies the first axiom of countability and for every point $x \in V$ there is a countable sequence of different points $x_n \in V$ converging to x , hence $g(V) \subset M$. Next let x be an arbitrary point of M , then there exists a sequence $\{x_n\}$, $x_n \in M$, $x_n \rightarrow x$ such that $g(x_n) \in M$. From the continuity of g it follows that $g(x_n) \rightarrow g(x)$ and hence $g(x) \in M$.

Thus $g(M) \subset M$ and g restricted to M is a quasi-local homeomorphism of M into itself. From this follows easily

$$Q(H) \simeq Q(M), \text{ so } Q(H) \simeq S.$$

Corollary. *Let S be a semigroup with cancellation and identity element. Then there is a connected compact Hausdorff space H such that S is isomorphic to the semigroup $T(H)$ of all topological transformations of H into H . Moreover $T(H) = Q(H)$.*

4. Local homeomorphisms and open continuous mappings

Let S be the semigroup $\{e, a, b\}$ with identity element e and multiplication defined by $ab = ba = aa = bb = a$.

Let H be a Hausdorff space and $L(H)$ the semigroup of all local homeomorphisms of H into itself.

$O(H)$ will denote the semigroup of all open continuous mappings of H into H .

Theorem 4. *There is no Hausdorff space H such that S is isomorphic to $L(H)$.*

Proof. Let S be isomorphic to $L(H)$. Then $L(H) = \{\varepsilon, f, g\}$ with ε the identity mapping and f and g local homeomorphisms such that $fg = gf = ff = gg = g$. Let A be the subset of H such that for each $a \in A$ $f(a) = g(a)$. Then A is closed. $A \neq H$ and $A \neq \emptyset$ since for each point $b \in f(H)$ we have $f(b) = g(b)$. We now prove that A is open. Let $p \in \overline{H \setminus A}$, $p \in A$. Let O be a neighbourhood of $f(p) = g(p)$ such that f is a homeomorphism on O .

Let V be a neighbourhood of p such that $f(V) \subset O$ and $g(V) \subset O$. Since $p \in \overline{H \setminus A}$, there is a point $x \in H \setminus A$, $x \in V$. Then it follows that $f(x) \neq g(x)$ and both $f(x)$ and $g(x)$ are contained in O .

Since $ff = fg$ we have $f(f(x)) = f(g(x))$ and hence f is not one-to-one on O , a contradiction.

Thus A is open and closed.

Now let ϕ be the mapping defined by

$$\phi(x) = \begin{cases} x & \text{for } x \notin A \\ g(x) & \text{for } x \in A \end{cases}$$

It is clear that ϕ is a local-homeomorphism of H . Since $g(H) \subset f(H) \subset A$, we have $\phi \neq f$, $\phi \neq g$. Furthermore for each $x \notin A$ we have $f(x) \notin g(H)$, since otherwise $f(x) = g(y)$ and hence $gf(x) = g(x) = gg(y) = g(y) = f(x)$. Thus $g(H) \neq f(H)$. Since $\phi(A) = g(A) = g(H) \neq A$, we have $\phi \neq \varepsilon$. This however is contradictory to the fact that each local homeomorphism ϕ of H is contained in $L(H)$.

Theorem 5. *There is no Hausdorff space H such that S is isomorphic to $O(H)$.*

Proof. Let $O(H) = \{\varepsilon, f, g\}$ with ε the identity and f and g open continuous mappings such that $fg = gf = ff = gg = g$. If $A = \{x \mid x \in H, f(x) = g(x)\}$, then $A \neq \emptyset$ and A is closed. Furthermore $g(H) \subset f(H) \subset A$, $f(H)$ and $g(H)$ open. Let $p \in \overline{H \setminus A}$, $p \in A$, then $f(p) = g(p) \in g(H)$ and hence there is an open set V , $p \in V$ such that $f(V) \subset g(H)$. Let $x \in H \setminus A \cap V$. Then $f(x) \in g(H)$ and hence $f(x) = g(y)$. Thus $g(f(x)) = g(x) = g(g(y)) = g(y) = f(x)$. From this it follows that $x \in A$, a contradiction.

The set $A = \{x \mid x \in H, f(x) = g(x)\}$ is an open and closed set. In the same way as in the proof of Theorem 4 we now can construct an open continuous mapping ϕ such that $\phi \notin O(H)$.

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